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DIFFUSION IN LAMINAR PIPE FLOW

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Abstract—A combination of analytical reasoning and experimental observation is used to investigate the spreading of a solute that has been injected in fully-developed, laminar pipe flow. The results indicate that diffusion in laminar flow depends very much upon the magnitude of a dimensionless parameter, ε, that is analogous to the reciprocal of the Péclét number of heat transfer. Most of the previous work in this area appears to apply for relatively large values of ε, and perturbation methods are used to obtain some solutions for relatively small values of ε.

NOMENCLATURE

C_0, first-order, outer concentration;
C_{F_0}, C_{R_0}, first order, inner concentrations;
c, concentration;
c_0, maximum concentration at $t = 0$;
c_a, cross-sectional average concentration;
D, molecular diffusion coefficient;
K, one-dimensional dispersion coefficient;
L, initial length of dye slug;
l, length over which $\partial c_a/\partial z$ is appreciable;
r, radial coordinate;
R, pipe radius;
t, time;
U, maximum velocity;
$u(x, y)$, longitudinal velocity distribution;
x, y, lateral coordinates;
z, axial coordinate.

Greek symbols

ε, D/UR;
ξ, ξ', τ, τ', coordinates for the inner problems.

INTRODUCTION

A SOLUTE injected into the fully-developed, laminar pipe flow depicted in Fig. 1 will disperse throughout the flow because of (1) diffusion, which results from relative motion between molecules of the solute and fluid, (2) convection, which transports the solute downstream and spreads it as a result of the non-uniform velocity distribution and (3) gravitational effects, which are the result of density differences between the fluid and injected solute. Density differences will be considered negligible herein, so that axi-symmetric diffusion in the laminar pipe flow of Fig. 1 can be described mathematically by solutions to the following equation (Taylor [1]):

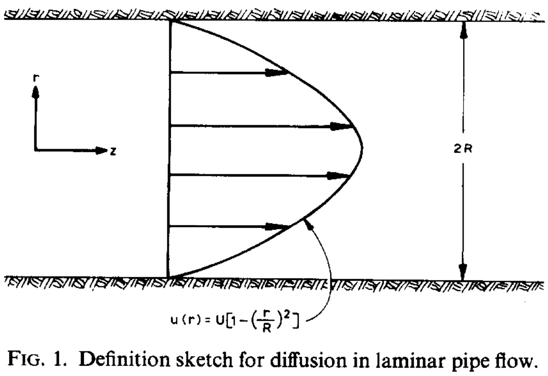

FIG. 1. Definition sketch for diffusion in laminar pipe flow.

$$D\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right)+\frac{\partial^2 c}{\partial z^2}\right] = U\left[1-\left(\frac{r}{R}\right)^2\right]\frac{\partial c}{\partial z}+\frac{\partial c}{\partial t}. \quad (1)$$

In equation (1), D = molecular diffusion coefficient, U = maximum, center-line velocity, R = pipe radius, c = solute concentration, r and z = cylindrical coordinates and t = time. The left side of equation (1) describes spreading by molecular diffusion, and the right side describes spreading by convection.

Griffiths [2] reported in 1911 the experimental result that a drop of fluorescent tracer solution injected into water flowing very slowly through a capillary tube spread symmetrically outward from a point of maximum concentration and that this point of maximum concentration moved with the average discharge velocity of the flow. In 1953 Taylor [1] attempted to quantify Griffith's experimental observations by writing equation (1) in a form

$$K\frac{\partial^2 c_a}{\partial z^2} = U_a\frac{\partial c_a}{\partial z}+\frac{\partial c_a}{\partial t} \quad (2)$$

in which c_a = average concentration over the pipe cross section, U_a = average discharge velocity (a constant) and

$$K \equiv \frac{R^2U_a^2}{48D}. \quad (3)$$

Taylor [1] also placed the following restriction upon solutions to equations (2) and (3):

$$\frac{D}{UR} \gg \frac{1}{3.8^2}\frac{R}{l}. \quad (4)$$

The variable l is defined by Taylor [3] as "the longitudinal extent of the region in which $\partial c_a/\partial z$ is appreciable". Equations (2) and (3) suffer from the almost obvious limitations that they cannot hold in the two limits when either $D \to 0$ or $U_a \to 0$. The first limitation was recognized by Taylor [1] in equation (4), but the second limitation was not recognized until one year later when Taylor [3] replaced equation (4) with

$$\frac{1}{13.8} \gg \frac{D}{UR} \gg \frac{1}{8}\frac{R}{l}. \qquad (5)$$

Taylor [1] was able to obtain experimental verification of his solutions to equations (2) and (3) over a limited range of values for D/UR.

In 1956 Aris [4] integrated equation (1) throughout the tube to solve for moments of c. His method is without restriction upon the values of D/UR, although only macroscopic characteristics of the distribution of c can be obtained from his analysis. One of the most important results of this study was to modify Taylor's expression for K in equation (3) and, as a result, to replace equation (5) with a less restrictive condition. This same result can be obtained in a different way by integrating equation (1) over the cross section of the pipe and making use of the definition for an average concentration

$$c_a \equiv \frac{2}{R^2}\int_0^R cr\,\mathrm{d}r. \qquad (6)$$

Then, if use is made of the facts that c_a is independent of r, that $\partial c/\partial r$ vanishes at both $r = R$ and $r = 0$ and that $U = 2U_a$ for laminar flow, equation (1) can be put in the exact form

$$D\frac{\partial^2 c_a}{\partial z^2} + \frac{4U_a}{R^4}\frac{\partial}{\partial z}\int_0^R (c-c_a)r^3\,\mathrm{d}r = U_a\frac{\partial c_a}{\partial z} + \frac{\partial c_a}{\partial t}. \qquad (7)$$

Equation (7) was first obtained by Philip [5] for slightly different purposes. Equation (7) has certain similarities to the von Kármán integral equation of boundary-layer theory, and one suspects that a quite accurate, approximate differential equation for c_a might be obtained if a reasonably accurate expression for the variation of $(c-c_a)$ over the pipe cross section could be substituted into the definite integral. Taylor [1, 3] assumed, on the basis of a "quasi-steady" solution of equation (1) when written in a coordinate system that translates with the discharge velocity, U_a, that

$$(c-c_a) = \frac{R^2U_a}{4D}\frac{\partial c_a}{\partial z}\left[-\tfrac{1}{3} + \left(\frac{r}{R}\right)^2 - \tfrac{1}{2}\left(\frac{r}{R}\right)^4\right]. \qquad (8)$$

Substitution of equation (8) into equation (7) then gives equation (2) with K redefined as

$$K \equiv D + \frac{R^2U_a^2}{48D}. \qquad (9)$$

Taylor [1, 3] obtained equation (3) instead of equation (9) because he chose to ignore the second term on the left side of equation (1).

Equations (2) and (9) reduce to the correct equations in the limit as $U_a \to 0$, so that it might be expected that equation (5) could be replaced with

$$\infty \geqslant \frac{D}{UR} \gg \frac{1}{8}\frac{R}{l}. \qquad (10)$$

Indeed, Aris [4] implies this when he states, on the basis of his analysis by moments, that equation (9) "is true without any restriction on the value of D/UR, or on the distribution of solute". A restriction similar to the one on the right side of equation (10) still exists, however, since Aris [4] obtains equation (9) as a result only after letting time, and therefore, l, become large. Also, equations (2) and (9) do not reduce to the correct equations as $D \to 0$, so that the results of Taylor and Aris must be invalid for small enough values of D/UR. Of particular importance to this study is the observation that equations (2) and (9) will give an accurate solution for c_a only when Taylor's approximation for $(c-c_a)$ in equation (8) is reasonably accurate. It will be seen later that equation (8) is an extremely poor approximation for very small values of D/UR, which also implies that a restriction similar to the RHS of equation (10) must be placed upon equations (2) and (9).

Bailey and Gogarty [6] obtained numerical solutions of equation (1) by neglecting, as did Taylor [1, 3], the term $\partial^2 c/\partial z^2$. Then they compared their numerical solutions with one of Taylor's solutions and with some experimental results of their own. All of their work was carried out for about the same range of values of D/UR as Taylor's earlier work, and they were able to obtain extremely close agreement with both Taylor's solution and Taylor's expression for $(c-c_a)$ in equation (8).

Lighthill [7] obtained some results for a slightly different aspect of the same problem. Lighthill, while working upon problems in blood flow, became interested in obtaining solutions in the region between the time of release of the solute and the time at which Taylor's and Aris's results become valid. During this period the solute has not yet had enough time to travel a distance of one pipe diameter by diffusion alone, and the actual concentration distribution is a perturbation of the solution with zero diffusion. Lighthill, after dropping the term $\partial^2 c/\partial z^2$ in equation (1), found an exact solution of the resulting approximate equation. This solution was sinusoidal in z, so that superposition and the Fourier transform were used to find a more general solution for the initial condition $c = f(z)$ at $t = 0$. The solution ignored the boundary condition $\partial c/\partial r = 0$ on the pipe wall since the solute did not have enough time to reach the pipe walls by molecular diffusion. The integral solution was too complicated to evaluate exactly, and the method of steepest descents was used to obtain an asymptotic approximation, in a fairly complicated form, for large values of $R^2/4Dt$.

Anathakrishnan, Gill and Barduhn [8] obtained finite-difference solutions of equation (1) for both large and small values of D/UR. This was done, however, by first introducing the transformation of variables

$$(y, x, \tau) = \left[\frac{r}{R}, \frac{z}{RU}\left(\frac{D}{t}\right)^{1/2}, \frac{Dt}{R^2}\right].$$

Thus, numerical solutions of the resulting equation are not suitable for taking the limit $D \to 0$, and the numerical results for small values of D/UR are of doubtful accuracy.

Finally the literature survey can be completed by noting that Gill [9] and Chatwin [10] have carried out calculations to define more precisely the range over which the solutions of Taylor and Aris are valid. Their results are only valid for a limited range of D/UR that is given approximately by equation (10), and satisfactory solutions for small values of D/UR have yet to be obtained.

PROBLEM DEFINITION AND FORMULATION

Let c_0 = the maximum value of c at $t = 0$. Then it becomes convenient to introduce the following dimensionless variables:

$$(c^*, r^*, z^*, t^*, \varepsilon) \equiv \left(\frac{c}{c_0}, \frac{r}{R}, \frac{z}{R}, \frac{Ut}{R}, \frac{D}{UR}\right). \quad (11)$$

If the dimensionless variables in equation (11) are substituted into equation (1), and if the asterisk superscript is omitted for notational convenience, equation (1) takes the following form:

$$\varepsilon\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right) + \frac{\partial^2 c}{\partial z^2}\right] = (1 - r^2)\frac{\partial c}{\partial z} + \frac{\partial c}{\partial t}. \quad (12)$$

The dimensionless parameter $\varepsilon \equiv D/UR$, which is analogous to the reciprocal of the Péclet number of heat transfer, can be characterized physically as a measure of the ratio of a diffusive velocity in motionless fluid to the maximum flow velocity at the pipe center. Taylor [1, 3] recognized, in equations (4) and (5), that $\varepsilon \equiv D/UR$ plays an important role in determining the behavior of solutions to equation (12). An idea of how much diffusion experiments are affected by changing the magnitude of ε can be obtained by looking at the experiments shown in Figs. 2–4. These experiments were conducted by submerging clear, rigid, plastic pipe (ID = 0.551 in = 1.4 cm) in water. Flow rates were controlled in Fig. 4 with a needle valve at the downstream end and in Fig. 3 by dripping water through a fine hypodermic needle at the downstream end. A slug of dye was introduced at the upstream end of the pipe, as shown in Fig. 2, and photographed later at points downstream, as shown in Figs. 3 and 4. The ratio R/l was estimated visually in Fig. 3 to be 2.92 $\times 10^{-3}$, and values of ε and Reynolds numbers were calculated as 1.37×10^{-3} and 0.643, respectively. Thus, conditions in Fig. 3 satisfy Taylor's inequality 5, and the experiment shown in Fig. 3 should be accurately described by Taylor's theory. Visual inspection of the experiment revealed that radial variations in c appeared small, that the dye slug appeared symmetric about its center and that the center of the dye slug moved at about the mean discharge velocity of the flow (U_a = 0.613 ft/h = 0.187 m/h). The experiment shown in Fig. 4 had a value of ε that was three orders of magnitude smaller ($\varepsilon = 1.47 \times 10^{-6}$, Reynolds number = 601, U_a = 1.91 in/s = 4.85 cm/s). Figure 4 shows sharp, almost discontinuous changes in dye concentration with zero concentration in a relatively

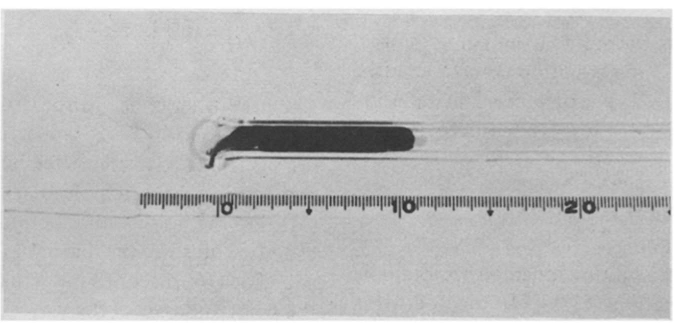

FIG. 2. Dye slug at start of the experiment.

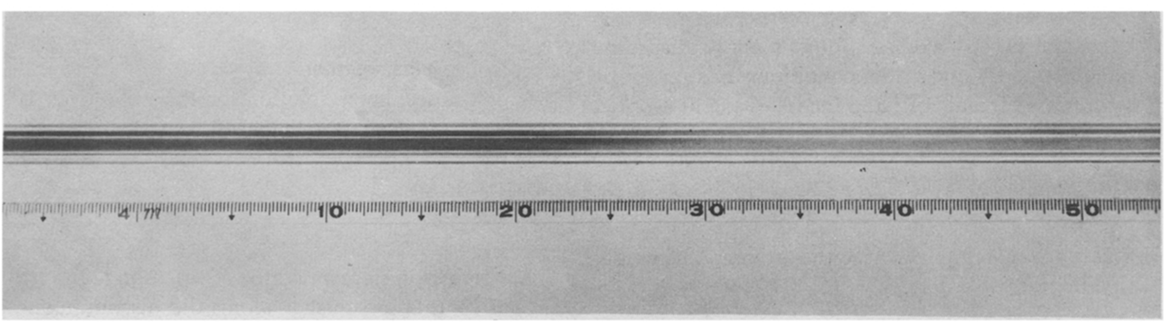

FIG. 3. Front of the dye slug at a distance of 614 radii downstream ($\varepsilon = 1.38 \times 10^{-3}$, Reynolds number = 0.643).

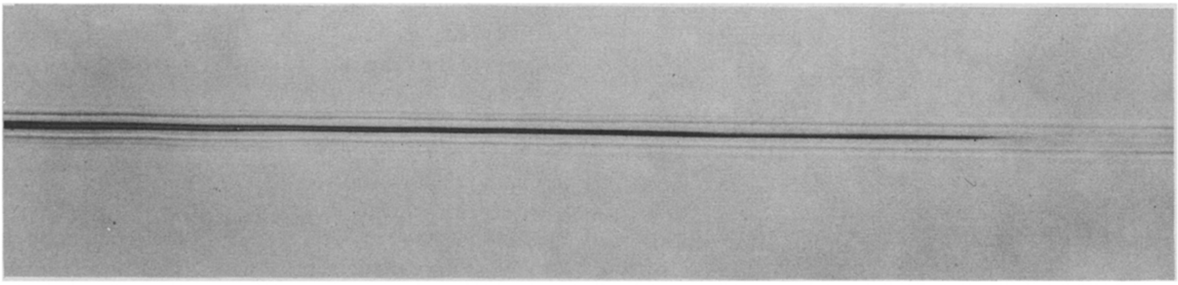

FIG. 4. Front of the dye slug at a distance of 1000 radii downstream ($\varepsilon = 1.47 \times 10^{-6}$, Reynolds number = 601).

large region near the pipe walls. Thus, equation (8) is a hopelessly inadequate description of the concentration across all cross sections, and Taylor's theory could not be applied. Convective velocities in Fig. 4 are very much greater than diffusive velocities, and the resulting concentration distribution is really a perturbation of the solution with zero diffusion. Hence, the techniques of singular perturbation theory, as described by Cole [11] and Van Dyke [12], will be used to solve equation (12) for flows similar to the one shown in Fig. 4, in which $\varepsilon \equiv D/UR \to 0$. Thus, these solutions will be valid in at least part of the region for which Taylor's results are invalid, as shown by equation (10).

Boundary and initial conditions for equation (12) will be taken as

$$\frac{\partial c(1,z,t)}{\partial r} = 0, \quad (0 < z, t < \infty) \tag{13}$$

$$c(r,z,0) = 1, \quad (0 < z < L, 0 \leqslant r < 1) \tag{14}$$
$$= 0, \quad (L < z < \infty, 0 \leqslant r < 1).$$

Equation (13) prevents the solute from being diffused through the pipe walls, and equation (14) is the initial condition shown in Fig. 2. The distance L in equation (14) is dimensionless and, thus, is measured in multiples of R.

PROBLEM SOLUTION

First-order, outer solution

The first-order, outer solution for small ε is found by substituting into equations (12)–(14) the following expansion, in which the omitted, higher-order terms vanish as $\varepsilon \to 0$:

$$c(r,z,t;\varepsilon) = C_0(r,z,t) + O(\sqrt{\varepsilon}). \tag{15}$$

If the limit $\varepsilon \to 0$ is taken, it is found that C_0 satisfies the same boundary and initial conditions as c [given by equations (13) and (14)] and the following first-order, partial differential equation:

$$(1-r^2)\frac{\partial C_0}{\partial z} + \frac{\partial C_0}{\partial t} = 0. \tag{16}$$

Equation (16) is equivalent to the ordinary differential equation

$$\frac{dC_0}{dt} = 0 \tag{17}$$

evaluated along the characteristic

$$\frac{dz}{dt} = (1-r^2). \tag{18}$$

Since r is treated as a parameter in equations (17) and (18), these equations can be integrated along a characteristic and the integration constants can be evaluated at a point $(z_1, t_1 = 0)$ upon the characteristic to obtain

$$C_0 = \text{constant} = 1, \quad (0 < z_1 < L, 0 \leqslant r < 1)$$
$$= 0, \quad (L < z_1 < \infty, 0 \leqslant r < 1) \tag{19}$$

along the characteristic curve

$$z - (1-r^2)t = \text{constant} = z_1. \tag{20}$$

Elimination of the constant parameter z_1 between equations (19) and (20) gives the first-order, outer solution in Eulerian coordinates as

$$C_0(r,z,t) = 1, \quad [0 < z-(1-r^2)t < L, 0 \leqslant r < 1]$$
$$= 0, \quad [L < z-(1-r^2)t < \infty, 0 \leqslant r < 1]. \tag{21}$$

The first-order concentration wave given by equation (21) is plotted in Fig. 5 and is seen to be simply the original slug of dye convected by the parabolic velocity distribution without molecular diffusion. This solution satisfies the initial and boundary conditions for c exactly, but, since the diffusion terms do not appear in equation (16), discontinuities in concentration occur at the front and rear of the wave. Thus, boundary layers form at these discontinuities as a result of molecular diffusion, and it is necessary to obtain first-order inner solutions along each of the two discontinuities.

First-order, inner solutions

A first-order, inner solution for the wave front is obtained by replacing t with the following, "magnified" variable

$$\tau \equiv \frac{t(1-r^2) - z + L}{\sqrt{\varepsilon}}. \tag{22}$$

The variables in the numerator of equation (22) were chosen so that τ vanishes along the discontinuous wave front, and the scaling factor in the denominator was chosen so that a non-trivial equation results when

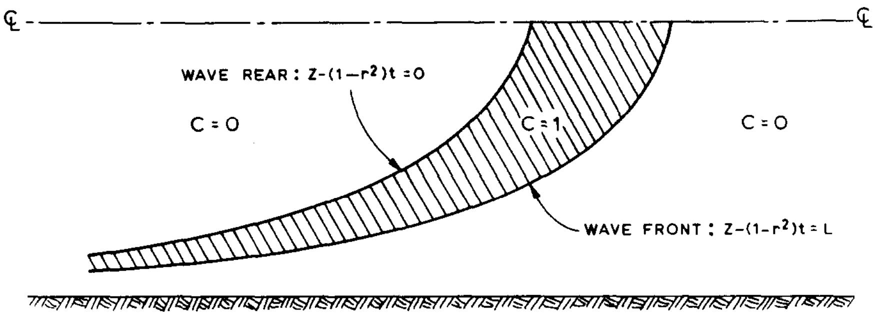

FIG. 5. A plot of the first-order, outer solution.

τ is introduced into equation (12) for t. Making the change of variables $(r, z, t) \to (r, z, \tau)$ in equation (12) gives

$$\varepsilon\left\{\left[\frac{1}{r}\frac{\partial}{\partial r} - \frac{2(z-L+\tau\sqrt{\varepsilon})}{(1-r^2)\sqrt{\varepsilon}}\frac{\partial}{\partial \tau}\right] \times\left[r\frac{\partial c}{\partial r} - \frac{2r^2(z-L+\tau\sqrt{\varepsilon})}{(1-r^2)\sqrt{\varepsilon}}\frac{\partial c}{\partial \tau}\right] + \left[\frac{\partial}{\partial z} - \frac{1}{\sqrt{\varepsilon}}\frac{\partial}{\partial \tau}\right]\left[\frac{\partial c}{\partial z} - \frac{1}{\sqrt{\varepsilon}}\frac{\partial c}{\partial \tau}\right]\right\} = (1-r^2)\frac{\partial c}{\partial z}. \quad (23)$$

An inner expansion for the wave front can be assumed in the form

$$c(r,z,\tau;\varepsilon) = C_{F_0}(r,z,\tau) + O(\sqrt{\varepsilon}). \quad (24)$$

Substitution of equation (24) into equation (23) and taking the limit as $\varepsilon \to 0$ gives

$$\left[1 + \frac{4r^2(z-L)^2}{(1-r^2)^2}\right]\frac{\partial^2 C_{F_0}}{\partial \tau^2} = (1-r^2)\frac{\partial C_{F_0}}{\partial z}. \quad (25)$$

Equation (25) can be simplified by setting

$$\xi \equiv \int_L^z \left[1 + \frac{4r^2(z-L)^2}{(1-r^2)^2}\right]\frac{\mathrm{d}z}{(1-r^2)} = \frac{(z-L)}{(1-r^2)}\left[1 + \frac{4r^2(z-L)^2}{3(1-r^2)^2}\right]. \quad (26)$$

Thus, replacing z with ξ in equation (25) gives

$$\frac{\partial^2 C_{F_0}}{\partial \tau^2} = \frac{\partial C_{F_0}}{\partial \xi}. \quad (27)$$

Matching, which can be done by writing $C_{F_0}(\xi, \tau)$ and $C_0(r, z, t)$ in terms of r, z, t and ε and r, ξ, τ and ε, respectively, taking the limits as $\varepsilon \to 0$ and equating the results, gives the boundary conditions

$$C_{F_0}(\xi, +\infty) = 1 \quad (28)$$

$$C_{F_0}(\xi, -\infty) = 0. \quad (29)$$

C_{R_0}

The correct initial condition is

$$C_{F_0}(0, \tau) = 1, \quad (\tau > 0)$$
$$= 0, \quad (\tau < 0). \quad (30)$$

The problem defined by equations (27)–(30) is a well-known problem in heat conduction that has the solution

$$C_{F_0} = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{\tau}{2\sqrt{\xi}}\right)\right] \quad (31)$$

in which erf is the error function.

A first-order, inner solution for the wave rear can be obtained in exactly the same way as for the wave front by defining

$$\tau' \equiv \frac{t(1-r^2) - z}{\sqrt{\varepsilon}} \quad (32)$$

$$\xi' \equiv \frac{z}{(1-r^2)}\left[1 + \frac{4r^2z^2}{3(1-r^2)^2}\right] \quad (33)$$

$$c(r, \xi, \tau) = C_{R_0}(\xi', \tau') + O(\sqrt{\varepsilon}). \quad (34)$$

Then C_{R_0} is found to be a solution of the following problem:

$$\frac{\partial^2 C_{R_0}}{\partial \tau'^2} = \frac{\partial C_{R_0}}{\partial \xi'} \quad (35)$$

$$C_{R_0}(+\infty, \xi') = 0 \quad (36)$$

$$C_{R_0}(-\infty, \xi') = 1 \quad (37)$$

$$C_{R_0}(\tau', 0) = 0, \quad (\tau' > 0)$$
$$= 1, \quad (\tau' < 0). \quad (38)$$

The solution to equations (35)–(38) is

$$C_{R_0} = \frac{1}{2}\left[1 - \mathrm{erf}\left(\frac{\tau'}{2\sqrt{\xi'}}\right)\right]. \quad (39)$$

First-order, composite solution

Typical plots of the inner and outer solutions, for fixed values of r and t, are shown in Fig. 6. A careful study of this plot suggests that a uniformly valid, composite solution can be defined in the following way:

$$\begin{aligned} C &= C_{F_0} - (1 - C_{R_0}), \quad [(1-r^2)t + L < z < \infty] \\ &= 1 - (1 - C_{R_0}) - (1 - C_{F_0}), \\ &\qquad [(1-r^2)t < z < (1-r^2)t + L] \\ &= C_{R_0} - (1 - C_{F_0}), \quad [0 < z < (1-r^2)t]. \end{aligned} \quad (40)$$

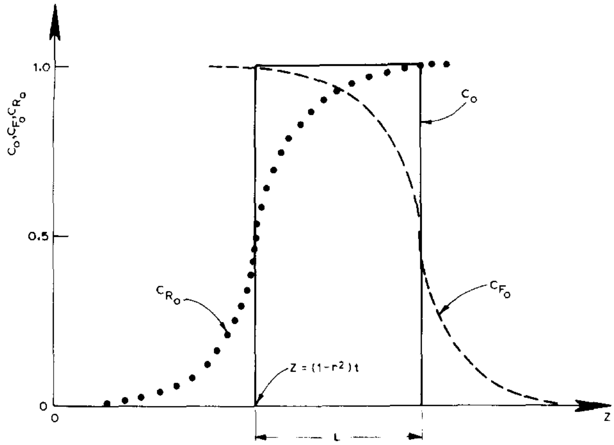

FIG. 6. Plots of the inner and outer solutions for particular values of r and t.

Hence, the uniformly valid, composite solution has the following very simple form:

$$c(r,z,t;\varepsilon) = \frac{1}{2}\left[\operatorname{erf}\left(\frac{\tau}{2\sqrt{\varepsilon}}\right) - \operatorname{erf}\left(\frac{\tau'}{2\sqrt{\xi'}}\right)\right] + O(\sqrt{\varepsilon}), \quad (0 < z, t < \infty, 0 \leqslant r < 1). \tag{41}$$

Careful examination of equation (41) and the analysis leading to equation (41) reveals that the boundary condition at the pipe wall, equation (13), is not satisfied. This implies that equation (41) will become invalid in regions where measureable concentrations occur next to the wall since equation (41) permits solute to diffuse through the wall of the mathematical model. When $\varepsilon \equiv D/UR$ is small, convective velocities near the pipe center are much larger than diffusive velocities and dye near the pipe center will be convected downstream before it has a chance to diffuse to the pipe walls. Thus, for small enough ε there is always a region near the dye front nose where zero concentrations exist next to the wall, as shown in Fig. 4. On the other hand, a small portion of the dye that was initially injected is soon left far behind because it is contained in a thin region next to the wall where convective velocities are as small, or smaller, than diffusive velocities. In this region equation (41) is invalid because dye does not diffuse through the experimental boundary. Experiments conducted by the writer, physical reasoning and calculations made with equation (41) all seem to indicate that, for small ε, the length of pipe over which wall concentrations are measureable grows with time but does not grow as rapidly as the length of pipe for which wall concentrations are negligible. For example, the band of zero concentration next to the pipe wall shown in Fig. 4 extended from the nose to a point just slightly downstream from where the dye was initially injected. Thus, equation (41) should be a valid description throughout a substantial portion of the region of interest.

Plots of concentration distributions calculated from equation (41) are shown at three different times in Figs. 7–9 for $\varepsilon = 10^{-4}$ and $L = 1$. Also shown are the outer solutions given by equation (10) which makes it relatively easy to see the effect of molecular diffusion in the problem. The values of both L and ε in Figs. 7–9 and Fig. 4 differ considerably. However, several similarities can be seen between the theoretical plots and the experimental photograph. Most notable are the relatively wide regions of zero concentration near the pipe wall. Also, careful observation of Fig. 4 reveals the core of zero concentration near the pipe center (behind the nose of dye), although this is obscured somewhat because the core is in the center of an annular region of dye. Experience in trying to calculate finite difference solutions in regions where concentrations vary as rapidly as those in Figs. 7–9 has convinced the writer of the futility of trying to verify equation (41) with numerical solutions. Experimental measurements might be better for this purpose, although they would be difficult to take accurately in such flows.

OTHER SOLUTIONS

It is quite possible to use the same perturbation methods to obtain solutions for other boundary or

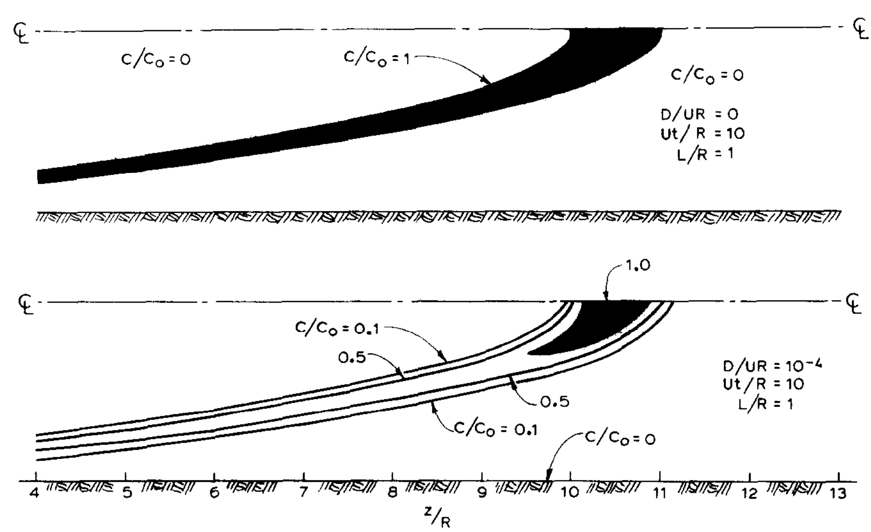

FIG. 7. Concentration distributions plotted from equations (21) and (41).

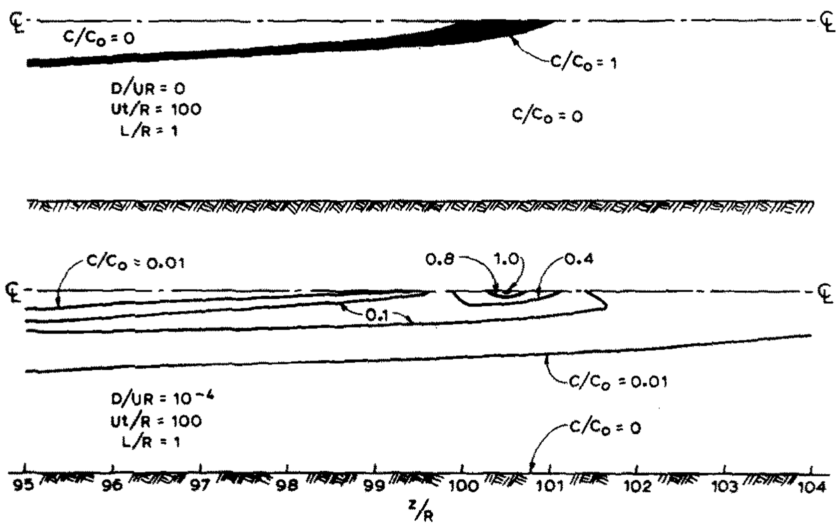

FIG. 8. Concentration distributions plotted from equations (21) and (41).

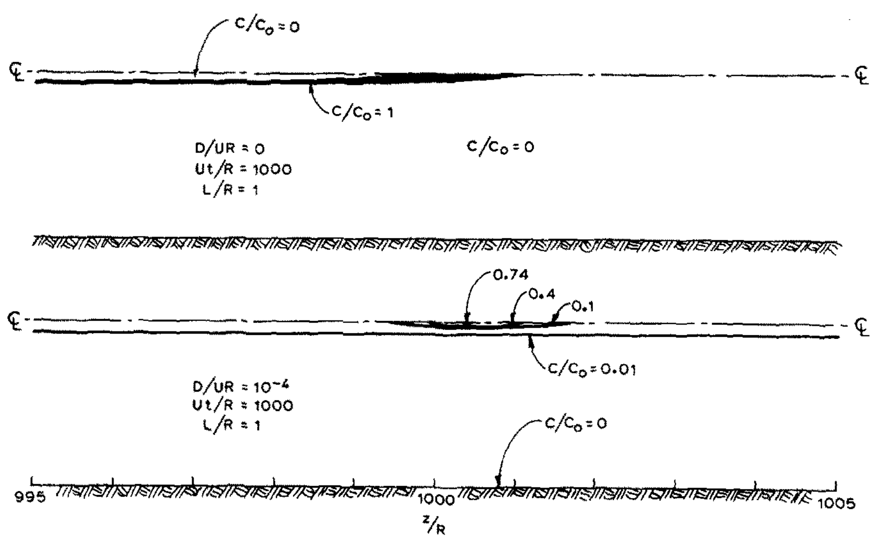

FIG. 9. Concentration distributions plotted from equations (21) and (41).

initial conditions. For example, the previous problem can be solved with the initial condition, given by equation (14), replaced by the following initial and boundary conditions:

$$c(r,z,0) = 0, \quad (0 \leqslant r < 1, 0 < z < \infty) \qquad (42)$$

$$c(r,0,t) = 1, \quad (0 \leqslant r < 1, 0 < t < \infty). \qquad (43)$$

The solution is

$$c(r,z,t;\varepsilon) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{\tau'}{2\sqrt{\xi'}}\right)\right] + O(\sqrt{\varepsilon}) \qquad (44)$$

in which τ' and ξ' are given by equations (32) and (33).

The more general problem for diffusion in uniform, laminar flow through any prismatic conduit, such as the conduit shown in Fig. 10, can be found by solving the following equation:

$$\varepsilon\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right) = u(x,y)\frac{\partial c}{\partial z} + \frac{\partial c}{\partial t}. \qquad (45)$$

The x, y and z coordinates in equation (45) have been made dimensionless with a characteristic lateral dimension of the conduit, and $u(x,y)$ has been made dimensionless with the maximum longitudinal flow velocity. The solution of equation (45) satisfying the intial and boundary conditions given by equations (13) and (14) is

$$c(x,y,z,t;\varepsilon) = \frac{1}{2}\left[\mathrm{erf}\left(\frac{\tau}{2\sqrt{\xi}}\right) - \mathrm{erf}\left(\frac{\tau'}{2\sqrt{\xi'}}\right)\right] + O(\sqrt{\varepsilon}) \qquad (46)$$

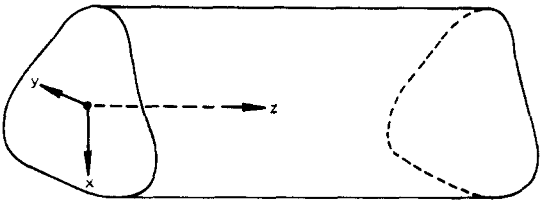

FIG. 10. Definition sketch for diffusion in laminar flow through any prismatic conduit.

in which τ, ξ, τ' and ξ' are defined as

$$\tau \equiv \frac{tu(x,y)-z+L}{\sqrt{\varepsilon}} \quad (47)$$

$$\xi \equiv \frac{(z-L)}{u(x,y)}\left\{1+\frac{\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right](z-L)^2}{3u^2(x,y)}\right\} \quad (48)$$

$$\tau' \equiv \lim_{L\to 0} \tau \quad (49)$$

$$\xi' \equiv \lim_{L\to 0} \xi. \quad (50)$$

Equations (46)–(50) reduce to equations (22), (26), (32), (33) and (41) when the cross section of the conduit is circular. Thus, they include solutions for flow through circular conduits, two-dimensional conduits and prismatic open channels as special cases. The solution satisfying the initial and boundary conditions given by equations (42) and (43) is

$$c(x,y,z,t;\varepsilon) = \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{\tau'}{2\sqrt{\xi'}}\right)\right] + O(\sqrt{\varepsilon}). \quad (51)$$

CONCLUSIONS

Solutions to diffusion problems in uniform, laminar flow are highly dependent upon the magnitude of the dimensionless parameter ε. Most of the earlier work, which is based upon the work by Taylor [7, 8], is valid for relatively large values of ε. The results obtained herein, which include solutions for flow through any prismatic conduit or open channel with several sets of boundary and initial conditions, are valid for relatively small values of ε. It is probable that a transition region also exists for which solutions are not yet available, and experimental work needs to be done in order to obtain more accurate limits for the ranges of ε in which these various solutions are valid.

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DIFFUSION DANS UN ECOULEMENT LAMINAIRE EN CONDUITE

Résumé—On utilise une combinaison d'un raisonnement analytique et d'une observation expérimentale pour étudier la dispersion d'un soluté injecté dans un écoulement pleinement développé et laminaire dans un tube. Les résultats montrent que la diffusion en écoulement laminaire dépend beaucoup de la valeur d'un paramètre adimensionnel ε qui est analogue à l'inverse du nombre de Péclet en transfert thermique. La plupart des travaux dans ce domaine concerne des valeurs relativement élevées de ε et on utilise ici la méthode des perturbations pour obtenir des solutions pour des valeurs faibles de ε.

DIE DIFFUSION IN LAMINARER ROHRSTRÖMUNG

Zusammenfassung—Eine Kombination aus analytischer Betrachtung und experimenteller Beobachtung wird zur Untersuchung des Ausbreitens eines lösungsfähigen Stoffes, der in eine voll ausgebildete laminare Rohrströmung eingeführt wird, herangezogen. Die Ergebnisse zeigen, daß die Diffusion in laminarer Strömung sehr stark von der Größe des dimensionslosen Parameters ε abhängt, der analog dem Kehrwert der Peclet–Zahl für den Wärmeübergang ist. Ein großer Teil der vorliegenden Arbeiten scheint sich nur auf relativ große Werte von ε zu beziehen; für relativ kleine Werte von ε erhält man einige Lösungen mit Hilfe von Störungsmethoden.

ДИФФУЗИЯ В ЛАМИНАРНОМ ПОТОКЕ В ТРУБЕ

Аннотация — С помощью аналитического метода и экспериментальных наблюдений исследуется процесс распространения раствора, вводимого в полностью развитый ламинарный поток в трубе. Результаты показывают, что на диффузию в ламинарном потоке сильно влияет величина безразмерного параметра ε, аналогичного обратному значению числа Пекле для процесса теплообмена. Повидимому, в большинстве предыдущих работ в этой области рассматривались относительно большие значения параметра ε. Для получения некоторых решений при относительно малых значениях ε используются методы возмущений.